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Best Rational Product Approximations of Functions. II*

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1. INTRODUCTION

The subject of this paper is the rational Chebyshev approximation of a continuous real valued function of several real variables. For simplicity our attention is focused on functions defined on the rectangle $D = [-1, 1] \times [-1, 1]$.

Let F be a continuous real valued function defined on D. For each $A = (a_{00}, \ldots, a_{nv_n}) \in E_{v_0+v_1+\cdots+v_n+n+1}$ and each $B = (b_{00}, \ldots, b_{mu_m}) \in E_{u_0+u_1+\cdots+u_m+m+1}$ define $\mathbb{R}$ to be the class of rational functions of the form

$$R(C; x, y) = \frac{P(A; x, y)}{Q(B; x, y)} = \frac{\sum_{i=0}^{n} \sum_{k=0}^{v_i} a_{ik} x^i y^k}{\sum_{j=0}^{m} \sum_{l=0}^{u_j} b_{jl} x^j y^l}, \tag{1}$$

where $C = (A; B)$ and $Q(B; x, y) > 0$ on D.

We formulate the problem of best rational Chebyshev approximation to F from $\mathbb{R}$ as follows. An element $R(C^*; x, y) \in \mathbb{R}$ is sought such that

$$\sup_{(x,y)\in D} |F(x, y) - R(C; x, y)|$$

is a minimum for $C = C^*$. $R(C^*; x, y)$ is called a *Chebyshev approximation* or a *best uniform approximation* to F.

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The following example demonstrates that a best uniform approximation from $\mathbb{R}$ to a given $F \in C[D]$ may not exist.

EXAMPLE 1. Let

$$F(x, y) = \begin{cases} \dfrac{(x+1)^2 + (y+1)^2}{x+y+2}, & \begin{matrix} -1 \leqslant x \leqslant 1 \\ -1 < y \leqslant 1 \end{matrix} \\ x+1 & \begin{matrix} -1 \leqslant x \leqslant 1 \\ y = -1 \end{matrix} \end{cases}$$

and consider the class of rational functions $\mathbb{R} = \{R(C; x)\}$, where $R(C; x)$ is (1) with $n = 2$, $m = 1$, $v_i = 2$ and $m_j = 1$, $i = 0, 1, 2$, $j = 0, 1$.

For each $\epsilon > 0$ define

$$R(C_\epsilon\,; x, y) = \frac{(x+1)^2 + (y+1)^2}{x + y + 2 + \epsilon}.$$

Then $R(C_\epsilon\,; x, y) \to F(x, y)$ uniformly on $[-1, 1] \times [-1, 1]$ as $\epsilon \to 0$. However, $F \notin \mathbb{R}$.

Loeb [6] showed that even when a continuous function F possesses a best approximation R, uniqueness could only occur if an appropriate vector space is a Haar space. The scarcity of multidimensional Haar spaces was demonstrated by Mairhuber [7].

The lack of existence or uniqueness of a best approximation and the accompanying computational difficulties have been a roadblock in the development of rational Chebyshev approximation. To overcome these problems, Henry and Brown [4] introduced best rational product approximations as follows.

We first define a class of rational functions. Let

$$R(C; x) = \frac{P(A; x)}{Q(B; x)} = \frac{\sum_{i=0}^{n} a_i x^i}{\sum_{j=0}^{m} b_j x^j} \not\equiv 0 \tag{2}$$

where $C = (A; B) = (a_0, a_1, \ldots, a_n; b_0, b_1, \ldots, b_m)$ satisfies

(i) $Q(B; x) > 0$ for all $x \in [-1, 1]$;

(ii) $P(A; x)$ and $Q(B; x)$ have no common factors other than constants; and

(iii) $\max_{j=0,\ldots,m} |b_j| = 1$.

DEFINITION 1. Let $\mathbb{R}(n, m)$ denote the class of rational functions consisting of all $R(C; x)$ as above satisfying conditions (i), (ii) and (iii), and the zero function which we allow the unique representation $R(C_0\,; x)$ where $C_0 = (0, 0,\ldots, 0; 1, 0,\ldots, 0)$.

DEFINITION 2. For each fixed $y \in [-1, 1]$ let F_y denote the univariate function defined for $-1 \leqslant x \leqslant -1$ by

$$F_y(x) = F(x, y). \tag{3}$$

Let

$$R(C(y); x) = \frac{P(A(y); x)}{Q(B(y); x)} \tag{4}$$

denote the best uniform approximation to $F_y(x)$ from $\mathbb{R}(n, m)$. That is $\sup_{-1 \leqslant x \leqslant 1} | F_y(x) - R(C; x)|$ is a minimum for $C = C(y) = (A(y); B(y))$. Note that for each $y \in [-1, 1]$ there exists a unique choice for $C(y)$ and thus it is a well defined function.

DEFINITION 3. Let the components $a_i(y)$ and $b_j(y)$ of the vector $C(y)$ be best approximated in the Chebyshev sense on $[-1, 1]$ by, respectively,

$$P_{\alpha_i}(y) = \sum_{k=0}^{v_i} a_{ik} y^k, \qquad i = 0, 1, \ldots, n \tag{5}$$

and

$$Q_{\beta_j}(y) = \sum_{l=0}^{u_j} b_{jl} y^l, \qquad j = 0, 1, \ldots, m \tag{6}$$

Then,

$$T(x, y) \equiv \frac{T_P(x, y)}{T_Q(x, y)} \doteq \frac{\sum_{i=0}^{n} P_{\alpha_i}(y)\, x^i}{\sum_{j=0}^{m} Q_{\beta_j}(y) x^j} = \frac{\sum_{i=0}^{n} \sum_{k=0}^{v_i} a_{ik} x^i y^k}{\sum_{j=0}^{m} \sum_{l=0}^{u_j} b_{jl} x^j y^l} \tag{7}$$

is called the *best rational product approximation*, with respect to y.

In general this approximation may not exist. If $C(y)$ is continuous then the approximations $P_{\alpha_i}(y)$ and $Q_{\beta_j}(y)$ exist. Furthermore we must insure the nonvanishing of the denominator $\sum_{j=0}^{m} Q_{\beta_j}(y)\, x^j$. This can be achieved by selecting the degrees u_j, $j = 0, 1, \ldots, m$ sufficiently large so that $\sum_{j=0}^{m} Q_{\beta_j}(y)\, x^j > 0$. This is possible since $\sum_{j=0}^{m} b_j(y)\, x^j > 0$ on $[-1, 1] \times [-1, 1]$.

With the continuity of $C(y)$ and sufficiently large degrees u_j, $j = 0, 1, \ldots, m$ the best rational product approximation exists and is unique. In Section 2 we consider sufficient conditions for the continuity of $C(y)$, various possible types of discontinuous and a method to combat the most common varieties.

Two alternatives to Definition 3 have been considered in the literature. If the variable x is fixed first, rather than y, then a product approximation with respect to x can be similarly defined. It is shown in [9] that in general these two approximations are distinct. A second alternative is to approximate the components $a_i(y)$ and $b_j(y)$ by rational functions rather than polynomials. This variation is considered in [4]. Either variant is handled analogously to that considered below.

2. The Continuity or Discontinuity of the Vector $C(y)$

We now consider sufficient conditions for the continuity of the parameter vector $C(y)$.

Definition 4. $R(C; x) = [P(A; x)/Q(B; x)] \in \mathbb{R}(n, m)$ is said to be of varisolvent degree

$$\mathscr{M}(C) = \begin{cases} 1 + \max\{n + \partial Q, m + \partial P\}, & R(C; x) \not\equiv 0 \\ 1 + n, & R(C; x) \equiv 0 \end{cases} \tag{8}$$

where ∂P and ∂Q denote the degrees of $P(A; x)$ and $Q(B; x)$, respectively.

Henry and Brown [4] proved the following theorem giving sufficient conditions for the continuity of the vector $C(y)$.

Theorem 1. *Suppose that for fixed $y^* \in [-1, 1]$, $\mathscr{M}(C(y^*)) = n + m + 1$. Then the function $C(y)$ is continuous at y^*.*

We improve this result somewhat.

Theorem 2. *Suppose that $\mathscr{M}(C(y))$ is constant on $(a, b) \subset [-1, 1]$. Then $C(y)$ is continuous on (a, b).*

Proof. One proof of this result is essentially identical with the proof of Theorem 1 presented in [4]. As an alternative consider the following:

Let $R(C(y); x) = [P(A(y); x)/Q(B(y); x)]$ denote the best approximation to F_y from $\mathbb{R}(n, m)$ and suppose that

$$\mathscr{M}(C(y)) \equiv k \qquad \text{for all} \quad y \in (a, b) \subset [-1, 1]$$

Let ∂P and ∂Q denote the degrees of $P(A(y); x)$ and $Q(B(y); x)$, respectively.

Case 1. $k = n + 1$ and $n < m$.

Then since

$$1 + \max\{n + \partial Q, m + \partial P\} \geqslant 1 + m > 1 + n, \tag{9}$$

we must have

$$R(C(y); x) \equiv 0 \qquad \text{for} \quad \begin{matrix} -1 \leqslant x \leqslant 1 \\ -1 \leqslant y \leqslant 1 \end{matrix}.$$

Thus $C(y) \equiv C_0 = (0, 0,..., 0; 1, 0,..., 0)$ for $-1 \leqslant y \leqslant 1$.

Case 2. Either $k = n + 1$ and $n \geqslant m$, or $k > n + 1$. Then $k \geqslant \max\{1 + n, 1 + m\}$.

The following lemma combined with Theorem 1 completes this proof.

LEMMA 1. *Let $R(C(y); x)$ and k be as in Case 2 and let*

$$N = k - m - 1 \quad \textit{and} \quad M = k - n - 1. \tag{10}$$

Then $R(C(y); x)$ is also the best approximation to F_y from the class $\mathbb{R}(N, M)$ and has degree $N + M + 1$ in that class. (Note that the degree of a rational function depends on which class of rationals we are considering.)

Proof. Since $\mathbb{R}(N, M) \subseteq \mathbb{R}(n, m)$, if $R(C(y); x) \equiv 0$, then $\mathscr{M}(C(y)) = n + 1$, $N = n - m$, $M = 0$ and $R(C(y); x)$ is also the best approximation to F_y in $\mathbb{R}(N, M)$. Furthermore, $R(C(y); x) \equiv 0$ has degree $1 + N = 1 + N + M$ in that class.

If $R(C(y); x) \not\equiv 0$ then

$$1 + n + \partial Q \leqslant k, \quad \text{or equivalently,} \quad \partial Q \leqslant k - n - 1 = M \tag{11}$$

and

$$1 + m + \partial P \leqslant k, \quad \text{or equivalently,} \quad \partial P \leqslant k - m - 1 = N, \tag{12}$$

with equality holding in at least one of (11) and (12).

Therefore $R(C(y); x) \in \mathbb{R}(N, M)$, and in this class $R(C(y); x)$ has degree

$$1 + \max\{N + \partial Q, M + \partial P\} = 1 + N + M \qquad \text{Q.E.D.}$$

Now Theorem 1 implies that

$$(a_0(y),\dots, a_N(y); b_0(y),\dots, b_M(y))$$

is continuous for $y \in (a, b)$ and thus

$$C(y) = (a_0(y),\dots, a_N(y), 0,\dots, 0; b_0(y),\dots, b_M(y), 0,\dots, 0)$$

is likewise continuous for $y \in (a, b)$.

We now present four examples of various possible types of discontinuities of $C(y)$. In each case $D = [-1, 1] \times [-1, 1]$ as usual.

EXAMPLE 2.

$$F(x, y) = \begin{cases} \dfrac{y + 1 + \frac{1}{2}x}{1 + \frac{1}{2}x}, & \begin{matrix} -1 \leqslant x \leqslant 1 \\ -1 \leqslant y < 0 \end{matrix} \\ 1 & \begin{matrix} -1 \leqslant x \leqslant 1 \\ 0 \leqslant y \leqslant 1 \end{matrix} \end{cases}$$

Then the best approximation to $F_y(x)$ from $\mathbb{R}(1, 1)$ is $R(C(y); x) \equiv F_y(x)$, and

$$C(y) = \begin{cases} (y+1, \tfrac{1}{2}; 1, \tfrac{1}{2}) & -1 \leqslant y < 0 \\ (1, 0; 1, 0) & 0 \leqslant y \leqslant 1. \end{cases}$$

EXAMPLE 3.

$$F(x, y) = \begin{cases} \dfrac{y + 1 + \frac{1}{2}x}{1 + \frac{1}{2}x}, & \begin{matrix} -1 \leqslant x \leqslant 1 \\ -1 \leqslant y < 0 \end{matrix} \\ 1 & \begin{matrix} -1 \leqslant x \leqslant 1 \\ y = 0 \end{matrix} \\ \dfrac{2y + 1 + \frac{1}{3}x}{1 + \frac{1}{3}x} & \begin{matrix} -1 \leqslant x \leqslant 1 \\ 0 < y \leqslant 1 \end{matrix} \end{cases}$$

Then the best approximation to $F_y(x)$ from $\mathbb{R}(1, 1)$ is again $R(C(Y); x) \equiv F_y(x)$ and

$$C(y) = \begin{cases} (y+1, \tfrac{1}{2}; 1, \tfrac{1}{2}), & -1 \leqslant y < 0 \\ (1, 0; 1, 0), & y = 0 \\ (2y+1, \tfrac{1}{3}; 1, \tfrac{1}{3}), & 0 < y \leqslant 1 \end{cases}$$

In both Examples 2 and 3 $C(y)$ has a property that will enable us to adjust the discontinuity at $y = 0$, and to define a modification of the best ratinoal product approximation. In particular if we let

$$C(y) = \begin{cases} (A_0(y); B_0(y)) & \text{for} \quad -1 \leqslant y < 0 \\ (A_1(y); B_1(y)) & \text{for} \quad 0 < y \leqslant 1 \end{cases} \tag{13}$$

then the functions A_0, A_1, B_0 and B_1 exist and are continuous on $[-1, 1]$. Furthermore

$$Q(B_0(y); x) > 0 \qquad \text{and} \qquad Q(B_1(y); x) > 0 \qquad \text{for} \quad -1 \leqslant y \leqslant 1.$$

In the following two examples this property does not hold and the adjustment will not be possible.

EXAMPLE 4.

$$F(x, y) = x + | y |.$$

Then the best approximation to $F_y(x)$ from $\mathbb{R}(0, 1)$ is

$$R(C(y); x) = \frac{\dfrac{y^2}{\sqrt{y^2+1}}}{1 - \dfrac{x}{\sqrt{y^2+1}}}, \qquad y \neq 0$$

$$R(C(0); x) \equiv 0.$$

Thus

$$C(y) = \begin{cases} \left(\dfrac{y^2}{\sqrt{y^2+1}}; 1, -\dfrac{1}{\sqrt{y^2+1}}\right), & y \neq 0 \\ (0; 1, 0), & y = 0 \end{cases}.$$

Note that with the convention of (13)

$$Q(B_0(y); x) = 1 - \frac{x}{\sqrt{y^2+1}}$$

and

$$Q(B_0(0); 1) = 0.$$

We next consider a more pathelogical example.

EXAMPLE 5.

$$F(x, y) = \frac{1 + \dfrac{1}{2}\left(y + \sin\dfrac{1}{y}\right) x}{1 + \left(\dfrac{1}{2}\sin\dfrac{1}{y}\right) x}, \qquad y \neq 0$$

$$F(x, 0) = 1$$

Then the best approximation to $F_y(x)$ from $\mathbb{R}(2, 1)$ is $R(C(y); x) \equiv F_y(x)$, and

$$C(y) = \begin{cases} \left(1, \dfrac{1}{2}\left(y + \sin\dfrac{1}{y}\right); 1, \dfrac{1}{2}\sin\dfrac{1}{y}\right), & y \neq 0 \\ (1, 0; 1, 0), & y = 0. \end{cases}$$

In each of the above examples $\mathscr{M}(C(y))$ is constant for $-1 \leqslant y < 0$ and for $0 < y \leqslant 1$. Other examples involving still other types of discontinuities of $C(y)$ can be constructed.

We now propose a method for combating the varieties of discontinuities encountered in Examples 2 and 3. For simplicity we consider only Example 3.

Define the functions

$$P^*(A^*(y); x) = \begin{cases} (y + 1 + \frac{1}{2}x)(1 + \frac{1}{3}x), & \begin{matrix} -1 \leqslant x \leqslant 1 \\ -1 \leqslant y \leqslant 0 \end{matrix} \\ (2y + 1 + \frac{1}{3}x)(1 + \frac{1}{2}x), & \begin{matrix} -1 \leqslant x \leqslant 1 \\ 0 < y \leqslant 1 \end{matrix} \end{cases}$$

and $Q^*(B^*(y); x) = (1 + \frac{1}{2}x)(1 + \frac{1}{3}x)$ on $D = [-1, 1] \times [-1, 1]$ and consider the rational function

$$R^*(C^*(y); x) = \frac{P^*(A^*(y); x)}{Q^*(B^*(y); x)}, \text{ on } D.$$

For each y such that $-1 \leqslant y \leqslant 1$, $R^*(C^*(y); x) \notin \mathbb{R}(1, 1)$. Thus this rational approximation is not of the originally chosen form. Furthermore, for each $y \in [-1, 0]$, $P^*(A^*(y); x)$ and $Q^*(B^*(y); x)$ have the common factor $1 + \frac{1}{3}x$. Similarly, for each $y \in (0, 1]$, P^* and Q^* have the common factor $1 + \frac{1}{2}x$. Thus, for each $y \in [-1, 1]$

$$R^*(C^*(y); x) \notin \mathbb{R}(2, 2).$$

However if we define

$$\mathbb{R}^*(n, m) = \left\{\frac{P}{Q} : \begin{array}{r}\text{degree of } P \leqslant n, \text{ degree of } Q \leqslant m \\ \text{and } Q > 0 \text{ on } [-1, 1]\end{array}\right\}.$$

then, $R^*(C^*(y); x) \in \mathbb{R}^*(2, 2)$ for each $y \in [-1, 1]$.

Furthermore

$$R^*(C^*(y); x) \equiv F(x, y) \quad \text{for} \quad \begin{cases} -1 \leqslant x \leqslant 1 \\ \qquad \text{and} \\ -1 \leqslant y < 0 \quad \text{or} \quad 0 < y \leqslant 1 \end{cases}$$

and

$$C^*(y) = \begin{cases} \left(y + 1, \frac{5}{6} + \frac{1}{3}y, \frac{1}{6}; 1, \frac{5}{6}, \frac{1}{6}\right), & -1 \leqslant y \leqslant 0 \\ \left(2y + 1, \frac{5}{6} + y, \frac{1}{6}; 1, \frac{5}{6}, \frac{1}{6}\right), & 0 < y \leqslant 1 \end{cases}$$

which is continuous on $[-1, 1]$.

The rational approximation R^* obtained as above is not in general best in the enlarged class of rational functions.

We now generalize this procedure. As usual, for each fixed $y \in [-1, 1]$ let $R(C(y); x)$ denote the best approximation to $F_y(x)$ from $\mathbb{R}(n, m)$. Suppose that $C(y)$ is continuous on $[-1, 1]$ except at $y = y^*$. Let

$$R(C(y); x) = \begin{cases} \dfrac{P(A_0(y); x)}{Q(B_0(y); x)}, & \begin{array}{l} -1 \leqslant x \leqslant 1 \\ -1 \leqslant y < y^* \end{array} \\ \dfrac{P(A_1(y); x)}{Q(B_1(y); x)}, & \begin{array}{l} -1 \leqslant x \leqslant 1 \\ y^* < y \leqslant 1 \end{array} \end{cases} \tag{14}$$

where $Q(B_0(y); x)$ and $Q(B_1(y); x)$ exist and are positive for $-1 \leqslant x \leqslant 1$, $-1 \leqslant y \leqslant 1$. Note that Examples 4 and 5 fail in this regard.

Now define

$$P^*(A^*(y); x) = \sum_{i=0}^{n+m} a_i^*(y)\, x^i$$

$$= \begin{cases} P(A_0(y); x)\, Q(B_1(y); x), & -1 \leqslant x \leqslant 1 \\ & -1 \leqslant y \leqslant y^* \\ P(A_1(y); x)\, Q(B_0(y); x), & -1 \leqslant x \leqslant 1 \\ & y^* < y \leqslant 1 \end{cases} \tag{15}$$

and

$$Q^*(B^*(y); x) = \sum_{j=0}^{2m} b_j^*(y)\, x^j = Q(B_0(y); x)\, Q(B_1(y); x) \text{ on } D \tag{16}$$

and

$$R^*(C^*(y); x) = \frac{P^*(A^*(y); x)}{Q^*(B^*(y); x)}, \text{ on } D. \tag{17}$$

We note some of the properties of the function $R^*(C^*(y); x)$.

Remark 1. $R^*(C^*(y); x) = R(C(y); x)$, on D, except possibly at $y = y^*$.

Remark 2. Suppose further that $A_0(y)$ is continuous on $[-1, y^*]$, $A_1(y)$ is continuous on $[y^*, 1]$, $B_0(y)$ and $B_1(y)$ are continuous on $[-1, 1]$ and

$$\lim_{y \to y^{*-}} R(C(y); x) = \lim_{y \to y^{*+}} R(C(y); x), \qquad \text{for} \quad -1 \leqslant x \leqslant 1. \tag{18}$$

Then $C^*(y) = (A^*(y); B^*(y))$ is continuous on $[-1, 1]$.

Proof. By (16), $B^*(y)$ is continuous on $[-1, 1]$. Equation (15), the continuity of $A_0(y)$ on $[-1, y^*]$ and the continuity of $B_1(y)$ on $[-1, 1]$ imply that $\lim_{y \to y^{*-}} A^*(y) = A^*(y^*)$. Equation (18) implies that

$$\lim_{y \to y^{*-}} P^*(A^*(y); x) = \lim_{y \to y^{*+}} P^*(A^*(y); x).$$

Thus $\lim_{y \to y^{*-}} A^*(y) = \lim_{y \to y^{*+}} A^*(y)$, and therefore $A^*(y)$ is continuous on $[-1, 1]$.

Furthermore we note that while $R(C(y); x) \in \mathbb{R}(n, m)$, in general $R^*(C^*(y); x) \in \mathbb{R}^*(n + m, 2m) - \mathbb{R}(n, m)$ for each $y \in [-1, 1]$.

Under the hypothesis of Remark 2 we define the modified best rational product approximation $T^*(x, y)$ by

$$T^*(x, y) = \frac{T_P^*(x, y)}{T_Q^*(x, y)} = \frac{\sum_{i=0}^{n+m} H_{\alpha_i}^*(y)\, x^i}{\sum_{j=0}^{2m} G_{\beta_j}^*(y)\, x^j},$$

when as usual we choose $H_{\alpha_i}^*$ and $G_{\beta_j}^*$ as best uniform approximations (either polynomial or rational) to a_i^* and b_j^*, respectively, on $[-1, 1]$, for $i = 0,..., n + m$ and $j = 0,..., 2m$.

This procedure can be extended to more than one point of discontinuity. Suppose that $-1 < y_1 < y_2 < \cdots < y_k < 1$ and that $C(y)$ is continuous on $[-1, 1]$ except at $y = y_i$, $i = 1, 2,..., k$. Let

$$R(C(y); x) = \begin{cases} \dfrac{P(A_0(y); x)}{Q(B_0(y); x)}, & \begin{matrix} -1 \leqslant x \leqslant 1 \\ -1 \leqslant y < y_1 \end{matrix} \\ \dfrac{P(A_1(y); x)}{Q(B_1(y); x)} & \begin{matrix} -1 \leqslant x \leqslant 1 \\ y_1 < y < y_2 \end{matrix} \\ \cdots\cdots & \cdots\cdots \\ \dfrac{P(A_k(y); x)}{Q(B_k(y); x)}, & \begin{matrix} -1 \leqslant x \leqslant 1 \\ y_k < y \leqslant 1 \end{matrix} \end{cases}$$

where $Q(B_i(y); x) > 0$ for $-1 \leqslant x \leqslant 1$, $-1 \leqslant y \leqslant 1$ and $i = 0, 1,..., k$.

Define

$$Q^*(B^*(y); x) = \prod_{j=0}^{k} Q(B_j(y); x)$$

$$\pi_i(x, y) = \prod_{\substack{j=0 \\ j \neq i}}^{k} Q(B_i(y); x), \qquad i = 0, 1,..., k$$

and

$$P^*(A^*(y); x) = \begin{cases} P(A_0(y); x)\, \pi_0(x, y), & -1 \leqslant y \leqslant y_1 \\ P(A_1(y); x)\, \pi_1(x, y), & y_1 < y \leqslant y_2 \\ \cdots\cdots & \cdots\cdots \\ P(A_k(y); x)\, \pi_k(x, y), & y_k < y \leqslant 1. \end{cases}$$

Finally, define the approximation

$$R^*(C^*(y); x) = \frac{P^*(A^*(y); x)}{Q^*(B^*(y); x)} \text{ on } D.$$

In general $R^* \in \mathbb{R}^*(n + km, (k + 1)m)$. Thus, for large values of k, R^* may have a very large number of parameters.

We can then define the modified best rational product approximation as before.

3. Computation

The computation of the best rational product approximation first requires the computation of $R(C(y); x)$ for $-1 \leqslant y \leqslant 1$. In general, this is not possible. Instead we often choose a finite set of points $\{y_i\}_{i=1}^{k}$ such that

$-1 \leqslant y_1 < y_2 < \cdots < y_k \leqslant 1$, and compute $R(C(y_i); x)$ for $i = 1, 2,..., k$ (see [9] algorithms 1 and 2).

Several methods for computing best rational approximations are discussed in Cheney and Southard [2]. These algorithms are most successful when a good initial guess at the best approximation is available. Whenever $C(y)$ is continuous and y_i and y_{i+1} are sufficiently close, $R(C(y_i); x)$ is a good approximation to $R(C(y_{i+1}); x)$.

Iterative procedures such as the Remez exchange algorithm require a good guess at a *characteristic point set* for the best rational approximation. Such a point set is defined in the following characterization theorem (see Rice [8], p. 80).

THEOREM 3 (Characterization). *$R(C; x) \in \mathbb{R}(n, m)$ is a best approximation to $F(x) \in C[-1, 1]$ if and only if there exist $\mathscr{M}(C) + 1$ points*

$$-1 \leqslant x_0 < x_1 < \cdots < x_{\mathscr{M}(C)} \leqslant 1$$

such that

$$| R(C; x_i) - F(x_i)| = \sup_{x \in [-1,1]} | R(C; x) - F(x)|, \qquad i = 0, 1,..., \mathscr{M}(C), \tag{19}$$

and

$$R(C; x_{i+1}) - F(x_{i+1}) = F(x_i) - R(C; x_i), \qquad i = 0, 1,..., \mathscr{M}(C) - 1.$$

Such a point set $\{x_i\}_{i=0}^{\mathscr{M}(C)}$ is called a characteristic point set *for the best approximation $R(C; x)$. Any point x_i satisfying* (19) *is called an extreme point for $R(C; x) - F(x)$.*

A characteristic point set for $R(C(y_i); x)$ often provides a good initial guess at a similar point set for $R(C(y_{i+1}); x)$, $i = 0, 1,..., \mathscr{M}(C)$. This was shown for product polynomial approximations in [9]. The proof of the rational analog is identical with the proof for the polynomial case. Thus we omit the details and state the following theorem:

THEOREM 4. *Given an $\epsilon > 0$ and $y^* \in [-1, 1]$, there exists a $\delta = \delta(\epsilon, y^*) > 0$ such that for $-1 \leqslant y \leqslant 1$ and $| y - y^* | < \delta$, any extreme point for $R(C(y); x) - F(x, y)$ is within ϵ distance of some extreme point for $R(C(y^*); x) - F(x, y^*)$.*

COROLLARY. *Suppose that there are exactly k extreme points for $R(C(y); x) - F(x, y)$ for each y such that $-1 \leqslant y \leqslant 1$. Then there exist k continuous functions $x_1(\cdot), x_2(\cdot),..., x_k(\cdot)$ such that for $-1 \leqslant y \leqslant 1$, $-1 \leqslant x_1(y) < x_2(y) < \cdots < x_k(y) \leqslant 1$ are the extreme points for $R(C(y); x) - F(x, y)$.*

In addition to knowing the approximate location of a set of characteristic points it is useful to have a good estimate for $\mathcal{M}(C(y))$ the degree of the best approximation. The following theorem shows that there is often a useful relation between $\mathcal{M}(C(y_i))$ and $\mathcal{M}(C(y_{i+1}))$.

THEOREM 5. *Given any $y^* \in [-1, 1]$ there exists a $\delta = \delta(y^*) > 0$, such that $-1 \leqslant y \leqslant 1$ and $|y - y^*| < \delta$ imply that*

$$\mathcal{M}(C(y)) \geqslant \mathcal{M}(C(y^*)). \tag{20}$$

Proof. If $R(C(y^*); x) \equiv 0$ then $\mathcal{M}(C(y^*)) = n + 1$. For any

$$y \in [-1, 1]\, \mathcal{M}(C(y)) \geqslant n + 1.$$

Now suppose that $R(C(y^*); x) \not\equiv 0$ and that the theorem is false. Then there exists a sequence $\{y_i\}$ such that

$$y_i \to y^* \qquad \text{as} \quad i \to \infty$$

and

$$\mathcal{M}(C(y_i)) < \mathcal{M}(C(y^*)).$$

Let $R(C(y_i); x) = P_i/Q_i$. Then as in Theorem 1 either

$$\frac{P_i}{Q_i} \equiv 0$$

or

$$\partial P_i \leqslant \mathcal{M}(C(y_i)) - m - 1 < \mathcal{M}(C(y^*)) - m - 1 \tag{21}$$

and

$$\partial Q_i \leqslant \mathcal{M}(C(y_i)) - n - 1 < \mathcal{M}(C(y^*)) - n - 1. \tag{22}$$

Next we note that

$$\rho(y) = \sup_{-1 \leqslant x \leqslant 1} |F_y(x) - R(C(y); x)|$$

is continuous for $-1 \leqslant y \leqslant 1$, and that

$$\begin{aligned}\rho(y^*) &\leqslant \sup_{-1 \leqslant x \leqslant 1} |F_{y^*}(x) - R(C(y_i); x)| \\ &\leqslant \rho(y_i) + \sup_{-1 \leqslant x \leqslant 1} |F_{y_i}(x) - F_{y^*}(x)|\end{aligned}$$

Therefore

$$\lim_{i \to \infty} \sup_{-1 \leqslant x \leqslant 1} |F_{y^*}(x) - R(C(y_i); x)| = \rho(y^*). \tag{23}$$

Furthermore, for all $R(C; x) \in \mathbb{R}(n, m)$ there exist positive constants K and L such that

$$\sup_{-1 \leqslant x \leqslant 1} | R(C; x)| \leqslant K$$

implies that

$$\| C \| = \max\{| a_i |: i = 0,..., n, | b_j |: j = 0,..., m\} < L$$

where $C = (a_0 ,..., a_n ; b_0 ,..., b_m)$ (see Rice [8] p. 75). Therefore (23) implies that $\{R(C(y_i); x)\}$ is a uniformly bounded sequence.

Hence there exists a subsequence $\{C(y_{i_\nu})\}$ converging to $\bar{C} \in E_{n+m+2}$. Let $R(\tilde{C}; x)$ denote the element in $\mathbb{R}(n, m)$ associated with $\bar{C}$. That is Rice [8, p. 77] shows the existence of a $\tilde{C}$ such that

$$\lim_{\nu \to \infty} R(C(y_{i_\nu}); x) = R(\tilde{C}; x)$$

except at possibly a finite number of points in $[-1, 1]$.

Let $R(\tilde{C}; x) = \tilde{P}/\tilde{Q}$. Then either

$$\frac{\tilde{P}}{\tilde{Q}} \equiv 0$$

or by (21) and (22)

$$\partial \tilde{P} < \mathscr{M}(C(y^*)) - m - 1 \tag{24}$$

and

$$\partial \tilde{Q} < \mathscr{M}(C(y^*)) - n - 1. \tag{25}$$

However, Eq. (23) and the uniqueness of best approximation in $\mathbb{R}(n, m)$ imply that $\tilde{C} = C(y^*)$. Thus

$$\frac{\tilde{P}}{\tilde{Q}} \not\equiv 0,$$

and

$$\mathscr{M}(\tilde{C}) = \mathscr{M}(C(y^*)).$$

But, by (24) and (25)

$$\mathscr{M}(\tilde{C}) = 1 + \max\{n + \partial\tilde{Q}, m + \partial\tilde{P}\} < \mathscr{M}(C(y^*)) \quad \text{(contradiction).}$$

4. Error Bounds

In this section we bound the quantity $\| F - T^* \|$ where $T^*(x, y)$ is the modified best rational product approximation to $F(x, y)$ on $D = [-1, 1] \times [-1, 1]$, and

$$\| G \| = \sup_{(x,y) \in D} | G(x, y)|.$$

Similar estimates for the quantity $\| F - T \|$ where $T(x, y)$ is the best rational product approximation to $F(x, y)$ on D are given in [4].

Suppose that $C(y)$ is continuous on $[-1, 1]$ except at $y_1 < y_2 < \cdots < y_k$.

Let $Y^* = [-1, 1] - \{y_1, y_2, ..., y_k\}$ and let $T^*(x, y) = [T_P{}^*(x, y)]/[T_Q{}^*(x, y)]$ be the modified best rational product approximation to F on D. As in Remark 1 of Section 2,

$$R^*(C^*(y); x) = R(C(y); x)$$

for

$$-1 \leqslant x \leqslant 1 \qquad \text{and} \qquad y \in Y^*$$

Furthermore, both

$$\max_{-1 \leqslant x \leqslant 1} | F_y(x) - R^*(C^*(y); x)|$$

and

$$\max_{-1 \leqslant x \leqslant 1} | F_y(x) - R(C(y); x)|$$

are continuous functions of $y \in [-1, 1]$, even though $C(y)$ is discontinuous at $y_1, ..., y_k$. Then

$$\begin{aligned} \| F - R^* \| &= \max_{-1 \leqslant y \leqslant 1} \{ \max_{-1 \leqslant x \leqslant 1} | F(x, y) - R^*(C^*(y); x)|\} \\ &= \max_{-1 \leqslant y \leqslant 1} \{ \max_{-1 \leqslant x \leqslant 1} | F_y(x) - R^*(C^*(y); x)|\} \\ &= \sup_{y \in Y^*} \{ \max_{-1 \leqslant x \leqslant 1} | F_y(x) - R^*(C^*(y); x)|\} \\ &= \sup_{y \in Y^*} \{ \max_{-1 \leqslant x \leqslant 1} | F_y(x) - R(C(y); x)|\} \\ &= \max_{-1 \leqslant y \leqslant 1} \{ \max_{-1 \leqslant x \leqslant 1} | F_y(x) - R(C(y); x)|\} \\ &= \max_{-1 \leqslant y \leqslant 1} E_{n,m}(F_y), \end{aligned} \tag{26}$$

where

$$E_{n,m}(g) = \inf\{ \max_{-1 \leqslant x \leqslant 1} | g(x) - R(C; x)| : R(C; x) \in \mathbb{R}(n, m)\} \tag{27}$$

$$R^* - T^* = \frac{P^*}{Q^*} - \frac{T_P{}^*}{T_Q{}^*} = \frac{(P^* - T_P{}^*)\, T_Q{}^* + (T_Q{}^* - Q^*)\, T_P{}^*}{Q^* T_Q{}^*}$$

Let

$$\epsilon_Q{}^* = \| Q^* - T_Q{}^* \|, \qquad \epsilon_P{}^* = \| P^* - T_P{}^* \|$$

and

$$\min_{\substack{-1 \leqslant x \leqslant 1 \\ -1 \leqslant y \leqslant 1}} | Q^*(B^*(y); x)| = m_Q{}^*.$$

Then

$$\| T_P^* \| \leqslant \| P^* \| + \epsilon_P^*, \qquad \| T_Q^* \| \leqslant \| Q^* \| + \epsilon_Q^*$$

and

$$\min_{\substack{-1 \leqslant x \leqslant 1 \\ -1 \leqslant y \leqslant 1}} | T_Q^*(x, y)| \geqslant m_Q^* - \epsilon_Q^*$$

where we assume $\epsilon_Q^* < m_Q^*$.

Therefore

$$\| R^* - T^* \| \leqslant \frac{\epsilon_P^*[\| Q^* \| + \epsilon_Q^*] + \epsilon_Q^*[\| P^* \| + \epsilon_P^*]}{m_Q^*(m_Q^* - \epsilon_Q^*)}. \tag{28}$$

Combining (26) and (28) we obtain

$$\begin{aligned} \| F - T^* \| &\leqslant \| F - R^* \| + \| R^* - T^* \| \\ &\leqslant \max_{-1 \leqslant y \leqslant 1} E_{n,m}(F_y) + \frac{\epsilon_P^*[\| Q^* \| + \epsilon_Q^*] + \epsilon_Q^*[\| P^* \| + \epsilon_P^*]}{m_Q^*(m_Q^* - \epsilon_Q^*)}. \end{aligned} \tag{29}$$

Furthermore, suppose that

$$P^*(A^*(y); x) = \sum_{i=0}^{n+km} a_i^*(y)\, x^i$$

and

$$T_P^*(x, y) = \sum_{i=0}^{n+km} H_{\alpha_i}^*(y)\, x^i$$

where $H_{\alpha_i}^*(y)$ is some best approximation (either polynomial or rational) to $a_i^*(y)$ on $[-1, 1]$, for $i = 0, 1,..., n + km$. Then,

$$\epsilon_P^* = \| P^* - T_P^* \| \leqslant \sum_{i=0}^{n+km} \max_{-1 \leqslant y \leqslant 1} | a_i^*(y) - H_{\alpha_i}^*(y)|.$$

Similarly,

$$\epsilon_Q^* = \| Q^* - T_Q^* \| \leqslant \sum_{j=0}^{(k+1)m} \max_{-1 \leqslant y \leqslant 1} | b_j^*(y) - G_{\beta_j}^*(y)|,$$

where $G_{\beta_i}^*(y)$ is some appropriate approximation to $b_j^*(y)$ on $[-1, 1]$, for $j = 0, 1,..., (k + 1)m$.

The above can be utilized to show that we can often obtain an arbitrarily good modified rational product approximation. Given any $\epsilon > 0$ an n and m may be selected to ensure that

$$\max_{-1 \leqslant y \leqslant 1} E_{n,m}(F_y) < \frac{\epsilon}{2}$$

(see [9], p. 444).

We then choose the approximations $H^*_{\alpha_i}$, $i = 0, 1,..., n + km$, and $G^*_{\beta_j}$, $j = 0, 1,..., (k + 1)m$ sufficiently well so that

$$\frac{\epsilon_P^*[\| Q^* \| + \epsilon_Q^*] + \epsilon_Q^*[\| P^* \| + \epsilon_P^*]}{m_Q^*(m_Q^* - \epsilon_Q^*)} < \frac{\epsilon}{2}.$$

Then (29) implies that $\| F - T^* \| < \epsilon$. The above analysis establishes the following theorem.

THEOREM 6. *Let $F \in C[D]$. Given $\epsilon > 0$ there exists an $n(\epsilon)$ and $m(\epsilon)$ such that if $R(C(y); x)$ is the best rational product approximation to F_y on $[-1, 1]$, then*

$$\max_{-1 \leqslant x \leqslant 1} E_{n,m}(F_y) < \epsilon/2. \tag{30}$$

Suppose that $C(y)$ is continuous on $[-1, 1]$ except possibly at a finite number of points. If these discontinuities are such that the $R^*(C^*(y); x)$ of Section 2 exists, and if $T^*(x, y)$ is the modified best rational product approximation, then the best uniform approximations (either polynomial or rational) $H^*_{\alpha_i}(y)$ and $G^*_{\beta_j}(y)$ to $a_i^*(y)$ and $b_j^*(y)$, respectively, $i = 0, 1,..., n + km, j = 0, 1,...,$ $(k + 1)m$, may be selected to ensure that

$$\| R^*(C^*(y); x) - T^*(x, y)\| < \epsilon/2. \tag{31}$$

If inequalities (30) *and* (31) *are valid, then*

$$\| F - T^* \| < \epsilon.$$

We conclude this paper with an example.

EXAMPLE 6. Let $F_1(x, y) = | x | + | y | + 1$, $F_2(x, y) = | x | | y | + 1$. Again $D = [-1, 1] \times [-1, 1]$, and suppose that approximation is from $\mathbb{R}(n, n)$, n even, $n > 4$.

Then $R(C(y); x) = R(C; x) + | y | + 1$, where $R(C; x)$ is the best approximation from $\mathbb{R}(n, n)$ to $| x |$ on $[-1, 1]$. Thus

$$C(y) = [a_0 + b_0(| y | + 1),..., a_n + b_n(| y | + 1); b_0,..., b_n].$$

That is, $a_i(y) = a_i + b_i(| y | + 1)$, $b_i(y) = b_i$, $i = 0, 1,..., n$. We now suppose that $H_{\alpha_i}(y)$ and $G_{\beta_i}(y)$ are the best approximations from $\mathbb{R}(n, n)$ to $a_i(y)$ and $b_i(y)$, respectively.

Then

$$H_{\alpha_i}(y) = a_i + b_i(R(C; y) + 1),$$

and

$$G_{\beta_i}(y) = b_i, \qquad i = 0,..., n.$$

Therefore the best rational product approximation T_1 satisfies

$$\begin{aligned}
&| F_1(x, y) - T_1(x, y)| \\
&\quad \leqslant | F_y(x) - R(C(y); x)| + | R(C(y); x) - T_1(x, y) \\
&\quad \leqslant || x | - R(C; x)| + \left| \frac{\sum_{i=0}^{n} a_i(y)\, x^i}{\sum_{i=0}^{n} b_i(y)\, x^i} - \frac{\sum_{i=0}^{n} H_{\alpha_i}(y)\, x^i}{\sum_{i=0}^{n} b_i(y)\, x^i} \right| \\
&\quad \leqslant 3e^{-\sqrt{n}} + \left| \frac{\sum_{i=0}^{n} b_i(| y | - R(C; y))\, x^i}{\sum_{i=0}^{n} b_i x^i} \right|.
\end{aligned}$$

This implies that

$$\| F_1(x, y) - T_1(x, y)\| \leqslant 6e^{-\sqrt{n}}$$

Similar analysis shows that

$$\| F_2(x, y) - T_2(x, y)\| \leqslant 15e^{-\sqrt{n}},$$

where T_2 is the best rational product approximation to F_2 .

References

1. E. W. Cheney, "Introduction to Approximation Theory," McGraw-Hill, New York, 1966.
2. E. W. Cheney and T. H. Southard, A survey of methods for rational approximation, with particular reference to a new method based on a Formula of Darboux, *SIAMR*, **5** (1963), 219–231.
3. E. W. Cheney and H. L. Loeb, Generalized rational approximation, *SIAM J. Numer. Anal.* **1** (1964), 11–25.
4. M. S. Henry and J. A. Brown, Best rational product approximations to Functions, *JOAT* **3** (1973), 287–295.
5. M. S. Henry, Best Chebyshev composite approximations, presented at the National Science Foundation Regional Conference on Approximation Theory, University of California Riverside, June 15, 1972.
6. H. L. Loeb, Approximation by generalized rationals, *SIAM J. Numer. Anal.* **3** (1966), 34–55.
7. John C. Mairhuber, On Haar's theorem concerning Chebyshev approximation problems having unique solutions, *Proc. Amer. Math. Soc.* **7** (1956), 609–615.
8. John R. Rice, "The Approximation of Functions," Vol. 1, Addison-Wesley, Reading, Mass., 1964.
9. S. E. Weinstein, Approximations of Functions of several variables: product Chebyshev approximations, *JOAT* **2** (1969), 433–447.
10. S. E. Weinstein, Product approximations of functions of several variables, *SIAM J. Numer. Anal.* **8** (1971), 178–189.